The Monte-Carlo estimate of U is

$$\hat{U} = \frac{V}{N} \sum_{i=1}^{N} \|\boldsymbol{x}_i\|^2 , \tag{12}$$

where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ are N independent random vectors uniformly distributed in the Voronoi region of Λ.

To estimate $\operatorname{var} \hat{U}$, we first note that since the vectors $\boldsymbol{x}_i$ are independent and identically distributed, $\operatorname{var} \hat{U} = (V^2/N) \operatorname{var}\|\boldsymbol{x}\|^2$, where $\boldsymbol{x}$ is a single random vector with the same distribution as $\boldsymbol{x}_i$. Therefore, our estimate of $\operatorname{var} \hat{U}$, denoted by $\widehat{\operatorname{var}} \hat{U}$, is defined by

$$\widehat{\operatorname{var}} \hat{U} = (V^2/N) \widehat{\operatorname{var}}\|\boldsymbol{x}\|^2 . \tag{13}$$

Applying the standard unbiased variance estimator of $\operatorname{var}\|\boldsymbol{x}\|^2$

$$\widehat{\operatorname{var}}\|\boldsymbol{x}\|^2 = \frac{1}{N-1} \sum_{i=1}^{N} \left(\|\boldsymbol{x}_i\|^2 - \hat{U}/V \right)^2 \tag{14}$$

in (13) yields

$$\begin{aligned} \widehat{\operatorname{var}} \hat{U} &= \frac{1}{N(N-1)} \sum_{i=1}^{N} \left(V\|\boldsymbol{x}_i\|^2 - \hat{U} \right)^2 \\ &= \frac{1}{N-1} \left(\frac{V^2}{N} \sum_{i=1}^{N} \|\boldsymbol{x}_i\|^4 - \hat{U}^2 \right) \end{aligned} \tag{15}$$

or after normalization as in (4)

$$\hat{G} = \frac{\hat{U}}{nV^{1+2/n}}, \tag{16}$$

$$\widehat{\operatorname{var}} \hat{G} = \frac{\widehat{\operatorname{var}} \hat{U}}{(nV^{1+2/n})^2} . \tag{17}$$